It is of interest to note that an efficient algorithm for the numerical solution of (4) can be found in Ref. 6.

Linear Time-Varying Systems

The theory outlined in the previous sections is equally applicable to cases when the system (1) is time-varying and the terminal time in (2) is finite; that is, when (1) and (2) have the form

$$\dot{x} = A(t)x + B(t)u \tag{9}$$

$$J = \int_{t_0}^{t_f} e^{2\sigma t}(x'Qx + u'Ru)dt \tag{10}$$

Utilizing the arguments given previously with a slight modification, it is readily shown that the optimal control minimizing (10) subject to (9) is given by

$$u^* = -R^{-1}B'S(t)x \tag{11}$$

where the symmetric matrix $S(t)$ satisfies the matrix Riccati equation

$$\dot{S}(t) = -[A(t) + \sigma I]'S(t) - S(t)[A(t) + \sigma I] + S(t)B(t)R^{-1}B'(t)S(t) - Q,\ S(t_f) = 0 \tag{12}$$

On the other hand, using the theory of optimal linear regulator, the optimal control is

$$\hat{u} = -e^{-2\sigma t}R^{-1}B'(t)\hat{S}(t)z \tag{13}$$

where the symmetric matrix $\hat{S}(t)$ satisfies the following matrix Riccati equation:

$$\dot{\hat{S}}(t) = -A'(t)\hat{S}(t) - \hat{S}(t)A(t) + e^{-2\sigma t}\hat{S}(t)B(t)R^{-1}B'(t)\hat{S}(t) - e^{2\sigma t}Q$$

$$\hat{S}(t_f) = 0 \tag{14}$$

It can be demonstrated that (13) and (14) are identical to (11) and (12) by employing the nonsingular transformation $\hat{S}(t) = e^{2\sigma t}S(t)$. This identification also demonstrates the time invariance of the optimal control law as given by (3) and (4) for the time-invariant system when the terminal time approaches infinity.

An Example

Consider a third-order unstable system given by

$$A = \begin{bmatrix} 1 & 2 & -3.0 \\ -4 & 5 & -0.6 \\ 7 & 8 & -0.9 \end{bmatrix},\ B = \begin{bmatrix} 0.1 & 0 & 2 \\ 0.0 & 10 & 0 \\ 0.0 & 0 & 100 \end{bmatrix}$$

Taking the matrix $R = I$ and the matrix Q arbitrarily to be 0.1I with $\sigma = 0$, the eigenvalues of the resulting closed-loop system are -30.63 and $-4.58 \pm j2.33$. Using the same Q and R as before but with $\sigma = 3$, the eigenvalues of the resulting closed-loop system become -33.70 and $-9.90 \pm j2.30$. It can be seen that for a given set of weighting matrices Q and R, the design based on the exponentially time-weighted performance index does give a fast and well-damped system response. However, as pointed out in Ref. 1, although the use of exponentially time-weighted performance indices provides a satisfactory means of shifting the dominant eigenvalue of the system, the remaining modes of the system may be quite oscillatory. It appears that the applicability of the exponentially time-weighted performance indices is greatly enhanced when they are used in conjunction with a design technique such as the root-square locus.[7]

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Upwash Interference on an Oscillating Wing in Slotted-Wall Wind Tunnels

CHING-FANG LO*
ARO Inc., Arnold Air Force Station, Tenn.

Introduction

THE wall interference on a stationary wing in subsonic flow in a wind tunnel with ventilated walls has been studied extensively (see a summary in Ref. 1). However, the calculation of the interference on an oscillating airfoil has been limited because of the complexity of the problem. Experimental evidence indicates[2] that the interference effects on an oscillating airfoil may be large in slotted wall tunnels; however, calculations of this interference are limited to certain special cases.[2]

This paper presents the upwash interference on an oscillating wing in a slotted-wall tunnel for all frequencies. The formulation is based on the small-wing theory with a relationship between the steady acceleration and unsteady velocity potentials. An analytical solution is given for the upwash interference in a circular and in a rectangular tunnel with solid side walls.

Analysis

If the flow is oscillating with the angular frequency ω due to the harmonic motion of a wing, the perturbation potential may be written as $\phi = \phi_k(x,y,z)e^{i\omega t}$. The linearized equation for ϕ of a thin wing becomes

$$\beta^2 \frac{\partial^2\phi_k}{\partial x^2} + \frac{\partial^2\phi_k}{\partial y^2} + \frac{\partial^2\phi_k}{\partial z^2} - 2ikM^2\frac{\partial\phi_k}{\partial x} + k^2M^2\phi_k = 0 \tag{1}$$

where $k = \omega L/U$ is the reduced frequency and $\beta^2 = 1 - M^2$. U and M are freestream velocity and Mach number, respectively. All lengths are nondimensionalized by a tunnel characteristic length L.

From the definition of the acceleration potential

$$\psi_k = (\partial\phi_k/\partial x + ik\phi_k) \tag{2}$$

the inverse relation for ϕ_k can be obtained by integration with respect to x since ϕ_k vanishes far upstream at $x = -\infty$

$$\phi_k = \int_0^\infty e^{-ik\xi}\,\psi_k(x - \xi,y,z)d\xi \tag{3}$$

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* Research Engineer, Propulsion Wind Tunnel Facility. Member AIAA.

The model chosen for an oscillating wing is the counterpart of a horseshoe vortex in the steady case. The acceleration potential ψ_{km}, due to such a harmonically pulsating horseshoe vortex with a small span s, located at the origin is given[3] by

$$\psi_{km} = \frac{s\Gamma_k}{4\pi L}\frac{\partial}{\partial z}\frac{\exp\{(ik/\beta^2)[M^2x - M(x^2+\beta^2y^2+\beta^2z^2)^{1/2}]\}}{(x^2+\beta^2y^2+\beta^2z^2)^{1/2}} \tag{4}$$

If a modified acceleration potential is defined as

$$\Psi_k = \psi_k \exp(-ikM^2x/\beta^2),$$

the field equation, Eq. (1), may be simplified to

$$\beta^2\frac{\partial^2\Psi_k}{\partial x^2} + \frac{\partial^2\Psi_k}{\partial y^2} + \frac{\partial^2\Psi_k}{\partial z^2} + \frac{k^2M^2}{\beta^2}\Psi_k = 0 \tag{5}$$

The term of $(kM/\beta)^2$ in Eq. (5) may be neglected for low frequencies; the compressible flow problem may then be reduced to a corresponding incompressible flow by using the Prandtl-Glauert rule as in the steady flow[5] case.

Now let us restrict our attention to incompressible flow, $M = 0$. The disturbance acceleration potential, Eq. (4), reduces to

$$\psi_{km} = (s\Gamma_k/4\pi L)[z/(x^2+y^2+z^2)^{3/2}] \tag{6}$$

which is also derived in Ref. 4.

The perturbation potentials are considered to be composed of two parts: $\phi_k = \phi_{ki} + \phi_{km}$ and $\psi_k = \psi_{ki} + \psi_{km}$ where ϕ_{km} and ψ_{km} are due to the oscillating wing in free air and ϕ_{ki} and ψ_{ki} are the interference potentials induced by the tunnel walls. The differential equation for incompressible case, Eq. (1), in terms of ψ_{ki} is Laplace's equation

$$\nabla^2(\psi_{ki} + \psi_{km}) = 0 \text{ or } \nabla^2\psi_{ki} = 0 \tag{7}$$

The boundary condition of an ideal slotted wall represented by the equivalent homogeneous condition[6] in terms of ψ_k is

$$\psi_{ki} + F\partial\psi_{ki}/\partial n = -(\psi_{km} + F\partial\psi_{km}/\partial n) \tag{8}$$

where the nondimensional geometric slot parameter F is related to the open area ratio a/l as $F = (1/L)(l/\pi)\ln[\csc(\pi a/2l)]$ and $\partial/\partial n$ denotes differentiation along the outward normal to the wall. Equation (7) is the basic field equation associated with the boundary condition, Eq. (8), in the present investigation.

Circular Wind Tunnels

Consider a circular wind tunnel of radius R, which is chosen as the characteristic length, L. The solution of ψ_{ki} from Eqs. (7) and (8) is obtained by Fourier transform and given in terms of Bessel functions as

$$\psi_{ki} = \frac{s\Gamma_k}{2\pi^2}\frac{\sin\theta}{R}\int_0^\infty A(q)\cos xq\, I_1(\rho q)\, dq \tag{9}$$

where

$$A(q) = -q\frac{(1-F)K_1(q) - qFK_0(q)}{(1-F)I_1(q) + qFI_0(q)} \tag{10}$$

The interference velocity potential ϕ_{ki} can be determined by substituting Eq. (9) into Eq. (3). The upwash interference W_{ki} at the center of the tunnel $\rho = 0$ is given after some simplification by

$$W_{ki}\bigg|_{r=0} = \frac{1}{R}\frac{\partial\phi}{\partial z}\bigg|_{\rho=0} = \frac{2s\Gamma_k}{\pi R^2}[Re(k,x) + iIm(k,x)] \tag{11}$$

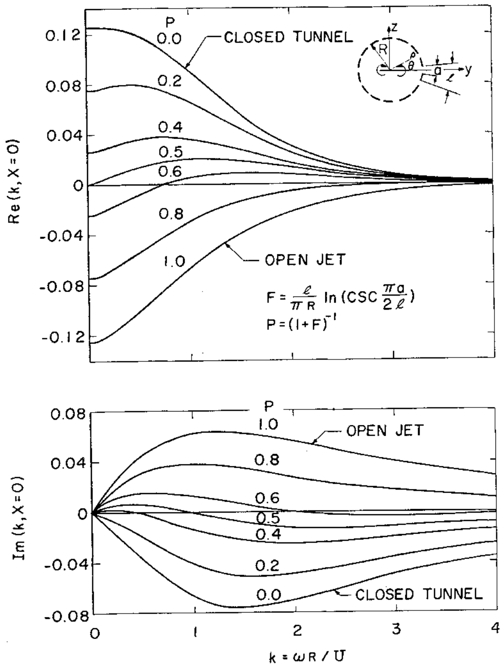

Fig. 1 Upwash interference at the plane of wing in circular slotted tunnels.

where

$$Re(k,x) = \frac{1}{8\pi}\left[\frac{\pi}{2}kA(k)\cos kx + \int_0^\infty \frac{q^2}{q^2-k^2}A(q)\sin qx\, dq\right] \tag{12}$$

$$Im(k,x) = \frac{1}{8\pi}\left[-\frac{\pi}{2}kA(k)\sin kx + k\int_0^\infty \frac{q}{q^2-k^2}A(q)\cos qx\, dq\right] \tag{13}$$

The real and imaginary parts of the upwash, Eq. (11), are shown in Fig. 1 at $x = 0$ for various values of the slotted parameter $P = (1+F)^{-1}$. The value of $P = 0.0$ corresponds to a closed tunnel and $P = 1.0$ an open jet. Taking $P = 0.0$ or $F \to \infty$ in Eq. (11), the closed tunnel solution[4] may be obtained.

Rectangular Tunnels with Solid Side Walls

Consider a rectangular wind tunnel with height $2h$ and width $2b$. The characteristic length L is chosen as b. The potential ψ_{ki} is solved by the image method in conjunction with Fourier transforms. The boundary condition at the closed side walls can be satisfied by introducing a horizontal row of images of the singularity, Eq. (6), along the plane $z = 0$. The interference acceleration potential ψ_{ki} can be obtained as

$$\psi_{ki} = \frac{1}{b}\frac{s\Gamma_k}{4\pi}\left\{\sum_{n=-\infty}^{\infty} n \neq 0\ \frac{z}{[x^2+(y+2n)^2+z^2]^{3/2}} + \int_0^\infty \sum_{m=0}^{\infty} jA(p_m)\sinh p_m z \cos m\pi y \cos xq\, dq\right\} \tag{14}$$

where

$$A(p_m) = \frac{(\lambda F p_m - 1)e^{-p_m\lambda}}{\sinh p_m\lambda + \lambda F p_m \cosh p_m\lambda}$$

$$\lambda = h/b,\ p_m = (q^2 + m^2\pi^2)^{1/2} \text{ and } j = \begin{cases} 1 & m = 0 \\ 2 & m \neq 0 \end{cases}$$

The upwash interference can be determined from Eqs. (14)

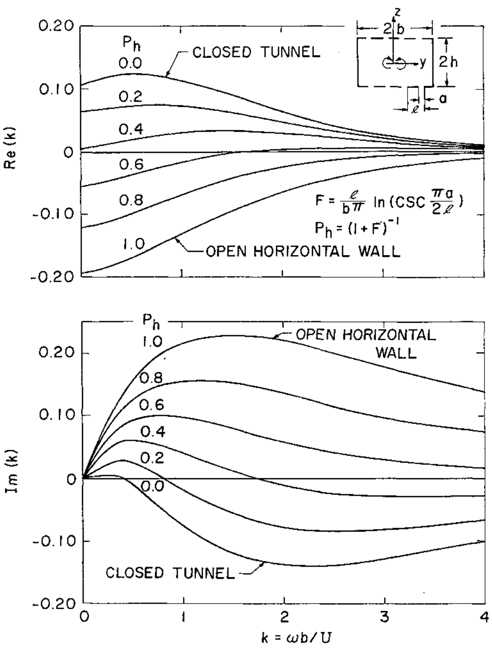

Fig. 2 Upwash interference at the plane of wing in rectangular tunnels with solid side walls of eight-to-width ratio 0.667.

and (3). At the plane of the wing, $x = y = z = 0$, the expression may be simplified to

$$W_{ki}|_0 = (1/b)\partial\phi_{ki}/\partial z|_0 = (s\Gamma_k/2bh)\{Re(k) + iIm(k)\} \quad (15)$$

where

$$Re(k) = \frac{1}{2\pi}\lambda\left\{\sum_{n=1}^{\infty}\frac{k}{n}K_1(2nk) + \frac{\pi}{2}\sum_{m=0}^{\infty} j \times (k^2 + m^2\pi^2)^{1/2} A[(k^2 + m^2\pi^2)^{1/2}]\right\}$$

$$Im(k) = \frac{k}{4}\lambda\left[\sum_{n=1}^{\infty}\frac{I_1(2nk) - L_{-1}(2nk)}{n} + \frac{4}{\pi}\sum_{m=0}^{\infty} j \int_0^{\infty} A(p_m)\frac{p_m}{q^2 - k^2}dq\right]$$

$L_{-1}(2nk)$ = Modified Struve function

The real and imaginary parts of the upwash interference relationship, Eq. (15), are plotted against reduced frequency for tunnels of height-to-width ratio $\lambda = 0.667$ in Fig. 2. The results of the static case derived in Ref. 1 may be obtained by taking the limit $k \to 0$ in Eq. (15).

Concluding Remarks

Results are given here for incompressible flow and all frequencies; however, they may be used for low frequencies in subsonic compressible flow with a minor modification, as in Eq. (5). All results are obtained for a small-wing model on the centerline of the tunnel. The expression for a model with an off-centerline position may be derived in a similar manner. Within the assumptions of linearized theory, solutions of any wing configuration may be obtained by superposition since it may be regarded as made up of "small wings," that is, lifting elements of area. The over-all corrections to forces and moments in wind tunnels as for general wings may be calculated[7] by the utilization of the upwash interference.

By examining the curves of the exact solutions given here, the validity of the approximation solution[2] in a power series of frequency up to the first order is within a rather narrow range of small values frequency, if the solution is extended to a slotted-wall tunnel.

The present results for slotted-wall tunnels provide information that will assist in choosing a tunnel wall configuration. For example, the optimum slot parameter of a circular cross section is about $P = 0.6$ from Fig. 1. For rectangular tunnel of height-to-width rat'o of 0.667 (Fig. 2), the optimum slot parameter 's in the range of $0.4 < P_h < 0.6$.

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A Further Note on Shock-Tube Measurements of End-Wall Radiative Heat Transfer in Air

Paul R. Knott,* Leland A. Carlson,† and Robert M. Nerem‡
The Ohio State University, Columbus, Ohio

SHOCK-tube measurements of the end-wall radiative heat transfer behind reflected shock waves in air have been previously reported in Ref. 1. These measurements were carried out at an initial shock-tube driven pressure of 1 mm Hg and at shock velocities ranging from 6.5 to 9.5 km/sec. A thin-film heat-transfer gage mounted behind a sapphire window located in the end wall of an arc-driven shock tube was used to measure the radiation in the wavelength region 0.17–6 μ.

In Ref. 1, comparisons were made between the experimental data and end-wall heat-transfer predictions based upon numerical solutions for the flowfield behind reflected shock waves in air. These equilibrium air solutions used several different models to account for radiative emission and absorption. Included were models for a gray gas as well as for a nongray gas. In the latter case, a two-step absorption co-

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* Research Assistant, Aeronautical and Astronautical Laboratory.

† Research Assistant, Aeronautical and Astronautical Laboratory. Member AIAA.

‡ Associate Professor, Department of Aeronautical and Astronautical Engineering. Associate Fellow AIAA.